

# Quantum graphs with two-particle contact interactions

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## Abstract

We construct models of many-particle quantum graphs with singular two-particle contact interactions, which can be either hardcore- or  $\delta$ -interactions. Self-adjoint realisations of the two-particle Laplacian including such interactions are obtained via their associated quadratic forms. We prove discreteness of spectra as well as Weyl laws for the asymptotic eigenvalue counts. These constructions are first performed for two distinguishable particles and then for two identical bosons. Furthermore, we extend the models to  $N$  bosons with two-particle interactions, thus implementing the Lieb-Liniger model on a graph.

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# 1 Introduction

This paper is the second in a series of papers developing models of interacting, non-relativistic many-particle systems on (compact) graphs. The underlying one-particle quantum graphs are well established models for a variety of problems in quantum mechanics. Interest in such one-particle models surged after Kottos and Smilansky [KS99b] had discovered that the eigenvalues of quantum Hamiltonians describing single particles on graphs possess the same correlations as eigenvalues of random hermitian matrices. Henceforth, quantum graphs proved to be very successful models in the area of quantum chaos [GS06] and beyond [EKK<sup>+</sup>08].

In our first paper [BK11] we introduced systems of two particles on compact metric graphs. We identified self-adjoint realisations of the two-particle Laplacian that describe singular two-particle interactions that are located in the vertices of the graph. In that context we analysed the Laplacians indirectly, by first constructing closed, semi-bounded quadratic forms and then identifying the self-adjoint operators that are uniquely associated with the quadratic forms. Using quadratic forms allowed us, furthermore, to prove that the Laplacians have compact resolvents and thus possess purely discrete spectra. Moreover, with a bracketing argument we were able to prove a Weyl law for the asymptotic eigenvalue count.

The goal of this paper now is to introduce two-particle interactions of a different kind, modelling short-range interactions in terms of singular contact interactions that are formally given by a Hamiltonian

$$H = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \alpha \delta(x - y) . \quad (1.1)$$

Here  $x$  and  $y$  are coordinates for the positions of the two particles on the edges of the graph and  $\alpha \in \mathbb{R}$  is a coupling parameter, such that  $\alpha > 0$  corresponds to repulsive interactions. Contact interactions of this kind play a prominent role in the description of Bose-Einstein condensates (BEC), leading to Gross-Pitaevskii equations, in plasma physics and in solid state physics (see, e.g., [LSSY05, CCG<sup>+</sup>11]). They are also of interest from a mathematical point of view: models of (an arbitrary number of) particles on the real line with Hamiltonian (1.1) are completely solvable in the sense that the eigenfunctions can be constructed explicitly, see [Yan67, AFK02, AFK04]. As in the one-particle case the connectivity of a graph, however, adds sufficient complexity to the problem to render it not solvable in that sense. Therefore, bosonic many-particle systems on graphs are expected to show generic many-particle properties of systems confined to one spatial dimension.

In this paper we shall follow our previous approach [BK11] in that we first introduce suitable quadratic forms, show that they are closed and semi-bounded, and then identify the corresponding self-adjoint operators. These operators shall always be two-particle Laplacians, however, with suitable domains that implement singular two-particle contact interactions. The resulting operators are then rigorous versions of the formal Hamiltonian (1.1). In order to achieve this the domains of the operators are required to contain jump conditions along diagonals  $x = y$  for derivatives of the functions in the domain. A

similar construction was already given by Harmer [Har07, Har08], who considered two particles on a star graph (with semi-infinite edges). Our construction works for any compact metric graph and allows for straight-forward generalisations to somewhat broader classes of singular contact interactions of which the  $\delta$ -type interactions (1.1) form a prominent subclass.

The paper is organised as follows: In Section 2 we review important properties of one-particle quantum graphs and introduce relevant concepts and notations that we are using in the following sections. Section 3 is then devoted to the construction of the contact interactions via quadratic forms for two distinguishable particles on a general compact, metric graph. In that context we encounter the problem of elliptic regularity [Dob05] in the same way as previously [BK11], suggesting an analogous solution for a similar class of boundary conditions. In Section 4 we first implement a bosonic realisation of a particle exchange symmetry and hence obtain a rigorous construction of the bosonic version of the operator (1.1). We then extend this construction to  $N$  bosons on a general compact graph, for which the equivalent to the formal operator (1.1) is

$$H = -\Delta_N + \alpha \sum_{i < j}^N \delta(x_i - x_j) . \quad (1.2)$$

Here  $x_1, \dots, x_N$  are the particle positions and  $-\Delta_N$  is the Laplacian in these  $N$  variables. Hence, the model we are investigating is an extension to quantum graphs of the well-known Lieb-Liniger model [LL63] that has been studied extensively in the context of BEC.

## 2 Preliminaries

Before introducing many-particle systems on graphs we briefly describe one-particle quantum graphs. They form the basis for the tensor-product construction of many-particle quantum systems on graphs.

The classical configuration space of a quantum graph is a compact metric graph, i.e., a finite graph  $\Gamma = (\mathcal{V}, \mathcal{E})$  with vertices  $\mathcal{V} = \{v_1, \dots, v_V\}$  and edges  $\mathcal{E} = \{e_1, \dots, e_E\}$ . The latter are identified with intervals  $[0, l_e]$ ,  $e = 1, \dots, E$ , thus introducing a metric on the graph, see [KS99b, KS99a, Kuc04, GS06] for details.

Functions  $F$  on the graph are collections of functions on the edges,

$$F = (f_1, \dots, f_E) , \quad \text{with} \quad f_e : [0, l_e] \rightarrow \mathbb{C} , \quad (2.1)$$

so that spaces of functions on  $\Gamma$  are (finite) direct sums of the respective spaces of functions on the edges. The most relevant space is the one-particle Hilbert space

$$\mathcal{H}_1 = L^2(\Gamma) := \bigoplus_{e=1}^E L^2(0, l_e) , \quad (2.2)$$

and all other spaces are constructed in a similar way.

The one-particle Hamiltonian is a Laplacian, acting as a second-order differentiation,

$$-\Delta_1 F = (-f_1'', \dots, -f_E'') , \quad (2.3)$$

on  $F \in C^\infty(\Gamma)$ . We here use the index to indicate that this is a one-particle Laplacian.

Domains of self-adjoint realisations of the Laplacian are characterised in terms of boundary conditions in the vertices. These require boundary values

$$F_{bv} := (f_1(0), \dots, f_E(0), f_1(l_1), \dots, f_E(l_E))^T \in \mathbb{C}^{2E} , \quad (2.4)$$

of functions and (inward) derivatives,

$$F'_{bv} := (f'_1(0), \dots, f'_E(0), -f'_1(l_1), \dots, -f'_E(l_E))^T \in \mathbb{C}^{2E} , \quad (2.5)$$

as well as projectors  $P$  and  $Q = \mathbf{1}_{2E} - P$  acting on the space  $\mathbb{C}^{2E}$  of boundary values, and self-adjoint endomorphisms  $L$  of  $\text{ran } Q \subset \mathbb{C}^{2E}$ .

The self-adjoint realisations of  $-\Delta_1$  can be identified via the quadratic forms that are uniquely associated with them [Kuc04].

**Theorem 2.1** (Kuchment). *The quadratic form*

$$Q_{P,L}^{(1)}[F] = \int_{\Gamma} |\nabla f|^2 dx - \langle F_{bv}, LF_{bv} \rangle_{\mathbb{C}^{2E}} = \sum_{e=1}^E \int_0^{l_e} |f'_e(x)|^2 dx - \langle F_{bv}, LF_{bv} \rangle_{\mathbb{C}^{2E}} , \quad (2.6)$$

with domain

$$\mathcal{D}_{Q^{(1)}} = \{F \in H^1(\Gamma); PF_{bv} = 0\} \quad (2.7)$$

is closed and bounded from below. The unique, self-adjoint and semi-bounded operator associated with this form is the one-particle Laplacian  $-\Delta_1$  with domain

$$\mathcal{D}_1(P, L) = \{F \in H^2(\Gamma); PF_{bv} = 0 \text{ and } QF'_{bv} + LQF_{bv} = 0\} . \quad (2.8)$$

A two-particle quantum system requires the tensor product of two one-particle Hilbert spaces,

$$\mathcal{H}_2 := \mathcal{H}_1 \otimes \mathcal{H}_1 . \quad (2.9)$$

For a quantum graph this means that

$$\mathcal{H}_2 = \left( \bigoplus_{e=1}^E L^2(0, l_e) \right) \otimes \left( \bigoplus_{e=1}^E L^2(0, l_e) \right) , \quad (2.10)$$

such that vectors  $\Psi \in \mathcal{H}_2$  are collections  $\Psi = (\psi_{e_1 e_2})$  of  $E^2$  functions defined on the rectangles  $D_{e_1 e_2} = (0, l_{e_1}) \times (0, l_{e_2})$ . Their disjoint union is denoted as

$$D_{\Gamma} = \dot{\bigcup}_{e_1 e_2} D_{e_1 e_2} , \quad (2.11)$$

so that one may view  $\mathcal{H}_2$  as

$$L^2(D_\Gamma) := \bigoplus_{e_1 e_2} L^2(D_{e_1 e_2}) . \quad (2.12)$$

We shall use a similar notation for other function spaces.

As a differential operator, the two-particle Laplacian acts as

$$(-\Delta_2 \Psi)_{e_1 e_2} = -\frac{\partial^2 \psi_{e_1 e_2}}{\partial x_{e_1}^2} - \frac{\partial^2 \psi_{e_1 e_2}}{\partial x_{e_2}^2} , \quad (2.13)$$

and hence has the same form as a Laplacian in  $\mathbb{R}^2$ . Defined on the domain  $C_0^\infty(D_\Gamma)$ , this operator is symmetric, but not self-adjoint.

Self-adjoint realisations of the two-particle Laplacians can either represent non-interacting particles, or introduce two-particle interactions via boundary conditions. A particular class of singular two-particle interactions that are localised in the vertices was established in [BK11]. Here we shall introduce two-particle contact interactions that are localised on the edges.

### 3 Contact interactions on a general compact graph

The interactions we have in mind are intended to model two point-like particles on a graph that interact when they hit each other, i.e., when they are located in the same position. This requires, in particular, that they are on the same edge. Hence, the subset of the two-particle configuration space  $D_\Gamma$  (2.11) where these interactions take place consists of the diagonals of the squares  $D_{ee}$ . Singular interactions require a dissection of the configuration space along these subspaces, and suitable matching conditions for functions and their derivatives along the boundaries introduced by the dissection. We therefore define the ‘dissected’ configuration space

$$D_\Gamma^* := \left( \bigcup_{e_1 \neq e_2} D_{e_1 e_2} \right) \dot{\bigcup}_e (D_{ee}^+ \dot{\bigcup} D_{ee}^-) , \quad (3.1)$$

where  $D_{ee}^+ = \{(x, y) \in D_{ee}; x > y\}$  and  $D_{ee}^- = \{(x, y) \in D_{ee}; x < y\}$ . Functions on  $D_\Gamma^*$  are denoted as  $\Psi = (\psi_{e_1 e_2})$ . The components  $\psi_{e_1 e_2}$  for  $e_1 \neq e_2$  are defined on  $D_{e_1 e_2}$ , whereas  $\psi_{ee} = (\psi_{ee}^+, \psi_{ee}^-)$  with  $\psi_{ee}^\pm$  defined on  $D_{ee}^\pm$ .

The two-particle Hilbert space  $\mathcal{H}_2$  (2.10) can then also be viewed as

$$L^2(D_\Gamma^*) = \left( \bigoplus_{e_1 \neq e_2} L^2(D_{e_1 e_2}) \right) \bigoplus_e (L^2(D_{ee}^+) \oplus L^2(D_{ee}^-)) . \quad (3.2)$$

Boundary values of functions  $\Psi \in H^1(D_\Gamma^*)$  are encoded in vectors

$$\Psi_{bv}(y) = (\psi_{e_1 e_2, bv}(y)) \quad \text{and} \quad \Psi'_{bv}(y) = (\psi'_{e_1 e_2, bv}(y)) . \quad (3.3)$$

We distinguish components with  $e_1 \neq e_2$  from those with  $e_1 = e_2$ , as in the latter case additional boundary values along diagonals have to be taken into account. More specifically, when  $e_1 \neq e_2$  we define

$$\psi_{e_1 e_2, bv}(y) := \begin{pmatrix} \sqrt{l_{e_2}} \psi_{e_1 e_2}(0, l_{e_2} y) \\ \sqrt{l_{e_2}} \psi_{e_1 e_2}(l_{e_1}, l_{e_2} y) \\ \sqrt{l_{e_1}} \psi_{e_1 e_2}(l_{e_1} y, 0) \\ \sqrt{l_{e_1}} \psi_{e_1 e_2}(l_{e_1} y, l_{e_2} y) \end{pmatrix} \quad \text{and} \quad \psi'_{e_1 e_2, bv}(y) := \begin{pmatrix} \sqrt{l_{e_2}} \psi_{e_1 e_2, x}(0, l_{e_2} y) \\ -\sqrt{l_{e_2}} \psi_{e_1 e_2, x}(l_{e_1}, l_{e_2} y) \\ \sqrt{l_{e_1}} \psi_{e_1 e_2, y}(l_{e_1} y, 0) \\ -\sqrt{l_{e_1}} \psi_{e_1 e_2, y}(l_{e_1} y, l_{e_2} y) \end{pmatrix}, \quad (3.4)$$

where  $y \in [0, 1]$ . When  $e_1 = e_2$  boundary values along the diagonals of the squares  $D_{ee}$  have to be added, including those for derivatives. Noting that the inward normal derivatives along the ‘diagonal’ part of the boundary of  $D_{ee}^\pm$  are

$$\psi_{ee, n}^\pm = \frac{\pm 1}{\sqrt{2}} (\psi_{ee, x}^\pm - \psi_{ee, y}^\pm), \quad (3.5)$$

we set

$$\psi_{ee, bv}(y) := \begin{pmatrix} \sqrt{l_e} \psi_{ee}^-(0, l_e y) \\ \sqrt{l_e} \psi_{ee}^+(l_e, l_e y) \\ \sqrt{l_e} \psi_{ee}^+(l_e y, 0) \\ \sqrt{l_e} \psi_{ee}^-(l_e y, l_e) \\ \sqrt{l_e} \psi_{ee}^+(l_e y, l_e y) \\ \sqrt{l_e} \psi_{ee}^-(l_e y, l_e y) \end{pmatrix} \quad \text{and} \quad \psi'_{ee, bv}(y) := \begin{pmatrix} \sqrt{l_e} \psi_{ee, x}^-(0, l_e y) \\ -\sqrt{l_e} \psi_{ee, x}^+(l_e, l_e y) \\ \sqrt{l_e} \psi_{ee, y}^+(l_e y, 0) \\ -\sqrt{l_e} \psi_{ee, y}^-(l_e y, l_e) \\ \sqrt{2l_e} \psi_{ee, n}^+(l_e y, l_e y) \\ \sqrt{2l_e} \psi_{ee, n}^-(l_e y, l_e y) \end{pmatrix}, \quad (3.6)$$

for  $y \in [0, 1]$ . Altogether, the vectors (3.3) of boundary values have  $n(E) := 4E^2 + 2E$  components.

As a next step we introduce the bounded and measurable maps  $P, L : [0, 1] \rightarrow \mathcal{M}(n(E), \mathbb{C})$  that are required to fulfil

1.  $P(y)$  is an orthogonal projector,
2.  $L(y)$  is a self-adjoint endomorphism on  $\ker P(y)$ ,

for a.e.  $y \in [0, 1]$ . We then introduce two bounded and self-adjoint operators,  $\Pi$  and  $\Lambda$ , on  $L^2(0, 1) \otimes \mathbb{C}^{n(E)}$ . They are defined to act as  $(\Pi\chi)(y) := P(y)\chi(y)$  and  $(\Lambda\chi)(y) := L(y)\chi(y)$  on  $\chi \in L^2(0, 1) \otimes \mathbb{C}^{n(E)}$ .

Our aim is to obtain self-adjoint realisations of the two-particle Laplacian  $-\Delta_2$ , see (2.13), that represent two-particle contact interactions and are extensions of  $-\Delta_{2,0}$  defined on the domain  $C_0^\infty(D_\Gamma^*)$ . In analogy to the one-particle case (2.8), as well as the case of singular interactions covered in [BK11], their domains should be given in the form

$$\mathcal{D}_2(P, L) := \{\Psi \in H^2(D_\Gamma^*); \ P(y)\Psi_{bv}(y) = 0 \text{ and} \\ Q(y)\Psi'_{bv}(y) + L(y)Q(y)\Psi_{bv}(y) = 0 \text{ for a.e. } y \in [0, l]\}. \quad (3.7)$$

In order to clearly distinguish the boundary conditions that induce contact interactions from other kinds of boundary conditions we rearrange the order of terms in the boundary

vectors. We first list, for each edge  $e$ , the lower two boundary values in (3.6), and then, for each pair  $(e_1, e_2)$ , either the four components in (3.4) or the upper four components of (3.6). That way one achieves a decomposition of the space of boundary values according to

$$\mathbb{C}^{n(E)} = V_{\text{contact}} \oplus V_{\text{vertex}} . \quad (3.8)$$

Here  $V_{\text{contact}}$ , with  $\dim V_{\text{contact}} = 2E$ , contains the boundary values (3.6) along diagonals, whereas  $V_{\text{vertex}}$ , with  $\dim V_{\text{vertex}} = 4E^2$ , contains the remaining boundary values (3.4) and (3.6), which are associated with vertices. A separation of contact interactions from any other boundary effects requires to choose  $P$  and  $L$  as block-diagonal with respect to the decomposition (3.8). From now on we assume this to be the case.

For the restriction of  $P$  and  $L$  to  $V_{\text{vertex}}$  we assume the same conditions as in [BK11]. For most purposes, however, it is sufficient to suppose that there are no two-particle interactions in the vertices. In [BK11] the non-interacting boundary conditions were characterised as follows: The restrictions of  $P$  and  $L$  to  $V_{\text{vertex}}$  are independent of  $y$  and block-diagonal with respect to a decomposition of  $V_{\text{vertex}}$  according to the index  $e_2$  in (3.3).

The restriction of  $P$  and  $L$  to  $V_{\text{contact}}$  should, first of all, be block-diagonal with respect to a decomposition of that space according to the edges in order to avoid ‘contact’ interactions across edges. Further restriction are not necessary, but we identify the following two cases as of particular interest because they correspond to a Hamiltonian of the form (1.1).

**Definition 3.1.** Let  $\alpha : [0, 1] \rightarrow \mathbb{R}$  be Lipschitz continuous. Then a contact interaction is said to be of

- (i)  *$\delta$ -type with (variable) strength  $\alpha$* , if  $\Psi \in H^2(D_\Gamma^*)$  is continuous across diagonals,

$$\psi_{ee}^+(l_e y, l_e y) = \psi_{ee}^-(l_e y, l_e y) , \quad (3.9)$$

and satisfies jump conditions for the normal derivatives,

$$\psi_{ee,n}^+(l_e y, l_e y) + \psi_{ee,n}^-(l_e y, l_e y) = \frac{1}{\sqrt{2}} \alpha(y) \psi_{ee}^\pm(l_e y, l_e y) , \quad (3.10)$$

- (ii) *hardcore type*, if it satisfies Dirichlet boundary conditions along diagonals.

We remark that contact interactions of the  $\delta$ -type can be seen as a rigorous realisation of a Hamiltonian

$$-\Delta_2 + \alpha(y) \delta(x - y) . \quad (3.11)$$

The case  $\alpha(y) > 0$  for all  $y \in [0, 1]$  corresponds to repulsive interactions and is the most relevant case for models of actual particles on a graph. Hardcore interactions follow from such a formal Hamiltonian in the limit  $\alpha \rightarrow \infty$ .

Following our intention to represent domains of two-particle Laplacians in the form (3.7) we have to choose the maps  $P$  and  $L$  in such a way that their restrictions to the edge- $e$  subspace of  $V_{\text{contact}}$  are

$$P_{\text{contact},e}(y) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (3.12)$$

and

$$L_{\text{contact},e}(y) = -\frac{1}{2} \alpha(y) \mathbf{1}_2 , \quad (3.13)$$

in order to generate  $\delta$ -type contact interactions. Hardcore interactions require the choice  $P_{\text{contact}} = \mathbf{1}$  and  $L_{\text{contact}} = 0$ .

Our approach to self-adjoint realisations of the Laplacian uses suitable quadratic forms, which are uniquely associated with these operators.

**Proposition 3.2.** *Assume that the maps  $P, L : [0, 1] \rightarrow M(n(E), \mathbb{C})$  are bounded and measurable. Then the quadratic form*

$$Q_{P,L}^{(2)}[\psi] = \langle \nabla \Psi, \nabla \Psi \rangle_{L^2(D_\Gamma^*)} - \int_0^1 \langle \Psi_{bv}(y), L(y) \Psi_{bv}(y) \rangle_{\mathbb{C}^{n(E)}} dy , \quad (3.14)$$

with domain

$$\mathcal{D}_{Q^{(2)}} = \{ \Psi \in H^1(D_\Gamma^*); P(y) \Psi_{bv}(y) = 0 \text{ for a.e. } y \in [0, 1] \} \quad (3.15)$$

is closed and semi-bounded.

*Proof.* The proof follows by using the same steps as in the corresponding proof in [BK11]. The only consideration that has to be added concerns the upper bound

$$\left| \int_0^1 \langle \Psi_{bv}(y), L(y) \Psi_{bv}(y) \rangle_{\mathbb{C}^{n(E)}} dy \right| \leq L_{\max} \|\Psi_{bv}\|_{L^2(0,1) \otimes \mathbb{C}^{n(E)}}^2 . \quad (3.16)$$

To estimate the right-hand side one requires the bound,

$$\|\Psi_{bv}\|_{L^2(0,1) \otimes \mathbb{C}^{n(E)}}^2 \leq K \left( \frac{2}{\delta} \|\Psi\|_{L^2(D_\Gamma^*)}^2 + \delta \|\nabla \Psi\|_{L^2(D_\Gamma^*)}^2 \right) , \quad (3.17)$$

to hold for all  $\delta \leq \delta_0$ , where  $K, \delta_0 > 0$ . The contribution from the rectangles  $D_{e_1 e_2}$  (with  $e_1 \neq e_2$ ) in the decomposition (3.1) can be dealt with as in [BK11] and is based on a result in [Kuc04]. For the triangles  $D_{ee}^\pm$  we note that close to the corners with angles  $\pi/4$  this method fails. However, one can always reflect functions  $\psi_{ee}^\pm$  across edges, define them on suitable squares and then apply the bound as before for the rectangles. The proof then continues as in [BK11].  $\square$

According to the representation theorem for quadratic forms (see, e.g., [Kat66]) there exists a unique self-adjoint and semibounded operator  $H$  with domain  $\mathcal{D}(H) \subseteq \mathcal{D}_{Q^{(2)}}$  that is associated with the quadratic form  $Q_{P,L}^{(2)}$ . It is not immediately clear, however, that the functions in  $\mathcal{D}(H)$  possess  $H^2$ -regularity. If this is the case we say, for short, that the quadratic form is *regular*. We note that a self-adjoint realisations of  $-\Delta_2$  with domain (3.7) would correspond to a regular form.

Under an additional (mild) assumption a regular quadratic form indeed leads to a two-particle Laplacian with domain (3.7).



**Proposition 3.3.** *Suppose that the map  $P$  is of class  $C^1$  and that the quadratic form  $Q_{P,L}^{(2)}$  with domain  $\mathcal{D}_{Q^{(2)}}$  is regular. Then the unique, self-adjoint and semibounded operator that is associated with this form is the two-particle Laplacian  $-\Delta_2$  with domain  $\mathcal{D}_2(P, L)$ .*

*Proof.* The proof can essentially be taken over verbatim from the corresponding proof in [BK11]. It is based on the representation theorem for quadratic forms, which implies that for each  $\Psi \in \mathcal{D}(H)$  there exists a unique  $\chi \in L^2(D_\Gamma^*)$  such that

$$Q_{P,L}^{(2)}[\Psi, \Phi] = \langle \chi, \Phi \rangle, \quad \forall \Phi \in \mathcal{D}_{Q^{(2)}}. \quad (3.18)$$

When  $\Phi \in C_0^\infty(D_\Gamma^*)$ , an integration by parts of (3.18) implies that  $H$  acts as a two-particle Laplacian  $-\Delta_2$ . In the general case of a  $\Psi \in \mathcal{D}_{Q^{(2)}}$  the integration by parts yields an additional boundary term,

$$- \int_0^1 \langle \Psi'_{bv}(y) + L(y)\Psi_{bv}(y), \Phi_{bv}(y) \rangle_{\mathbb{C}^{n(E)}} dy, \quad (3.19)$$

that is required to vanish. Following Lemma 3.13 in [BK11], which has an immediate generalisation to the present case, the set  $\{\Phi_{bv}; \Phi \in \mathcal{D}_{Q^{(2)}}\}$  is dense in  $\ker \Pi \subset L^2(0, 1)^{n(E)}$ . Hence,  $\Psi'_{bv} + \Lambda \Psi_{bv} \in \ker \Pi^\perp$ , or

$$Q(y)\Psi'_{bv}(y) + Q(y)L(y)\Psi_{bv}(y) = 0. \quad (3.20)$$

This condition finally implies that  $\mathcal{D}(H) = \mathcal{D}_2(P, L)$ .  $\square$

As mentioned above, the quadratic forms in Proposition 3.2 are not necessarily regular. Since our focus is on contact interactions of  $\delta$ - or hardcore-type, it is sufficient to consider these cases. However, as in [BK11] we have to add one additional assumption on the restrictions  $P_{vert}$  of the projectors  $P$  to  $V_{vertex}$ . Splitting  $V_{vertex}$  into the two subspaces spanned by the upper two and the lower two components of (3.4) as well as the upper two and middle two components of (3.6), respectively, we require  $P_{vert}$  to be block-diagonal with respect to this decomposition.

This then leads to the main result of this section.

**Theorem 3.4.** *In addition to the assumption made for the maps  $P$  and  $L$  above, suppose that  $P$  is of class  $C^3$  and  $L$  is Lipschitz continuous. Furthermore, for  $y \in [0, \epsilon_1] \cup [l - \epsilon_2, l]$  with some  $\epsilon_1, \epsilon_2 > 0$  assume that the restriction of  $P$  to  $V_{vertex}$  is diagonal with diagonal entries zero or one and, in the case of  $\delta$ -type interactions, that  $\alpha(y) = \alpha_0 \geq 0$  for those  $y$ . Then the quadratic form  $Q_{P,L}^{(2)}$  is regular.*

*Proof.* First note that it is enough to show regularity near the corners of  $D_{ee} = D_{ee}^+ \cup D_{ee}^-$  adjacent to the diagonal. The regularity away from the diagonal of  $D_{ee}$  as well as regularity in the rectangles  $D_{e_1 e_2}$  with  $e_1 \neq e_2$  was already established in [BK11]. In addition, the regularity along the diagonals in the interior of  $D_{ee}$  can be readily established using the same methods as in [BK11].

The assumptions made on  $P$  imply that on the edges of the squares  $D_{ee}$  the functions in  $\mathcal{D}_{Q(2)}$  satisfy either Dirichlet- or Neumann boundary conditions near the corners. Along diagonals we consider the projections

$$\begin{aligned}\psi_{ee,B}(x, y) &:= \frac{1}{2} [\psi_{ee}(x, y) + \psi_{ee}(y, x)] , \\ \psi_{ee,F}(x, y) &:= \frac{1}{2} [\psi_{ee}(x, y) - \psi_{ee}(y, x)] .\end{aligned}\tag{3.21}$$

The goal is to show that, close to the corners, both  $\psi_{ee,B}$  and  $\psi_{ee,F}$  are of class  $H^2$ . For that purpose one introduces suitable cut-offs that restrict the functions (3.21) to neighbourhoods of the corners. This eventually implies that  $\psi_{ee} \in H^2(D_{ee}^*)$ .

We recall the conditions (3.10) which imply that

$$\partial_n \psi_{ee,B}^\pm - \frac{\alpha}{2\sqrt{2}} \psi_{ee,B}^\pm = 0 ,\tag{3.22}$$

on the diagonal. Hence,  $\psi_{ee,B}^\pm$  satisfies (variable) Robin boundary conditions on the diagonal. By construction,  $\psi_{ee,F}^\pm$  vanishes on the diagonal so that near the corners of  $D_{ee}^\pm$  adjacent to the diagonal, where  $\alpha$  is supposed to be constant,  $\psi_{ee,B/F}^\pm$  satisfies a combination of Dirichlet-, Neumann- or standard Robin-boundary conditions. In all such cases regularity is well known to hold [Dau88, Gri85].  $\square$

One naturally expects the two-particle operators representing contact interactions to possess purely discrete spectra of the form  $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$  (i.e., eigenvalues are counted with their multiplicities and do not accumulate at any finite value). Moreover, the asymptotic distribution of eigenvalues, as given by the asymptotic behaviour of the eigenvalue-counting function

$$N(\lambda) := \#\{n; \lambda_n \leq \lambda\} ,\tag{3.23}$$

should follow a Weyl law. We shall now prove a Weyl law for repulsive contact interactions. This includes hardcore- and  $\delta$ -interactions with  $\alpha \geq 0$ . The general requirement is that  $L_{\text{contact}}$  is negative definite (compare (3.13)).

**Proposition 3.5.** *Let  $(-\Delta_2, \mathcal{D}_2(P, L))$  be a self-adjoint realisation of the two-particle Laplacian with repulsive contact interaction as described in Proposition 3.3. Then this operator has compact resolvent. In particular, its spectrum is purely discrete and only accumulates at infinity. Furthermore, the counting function (3.23) obeys the Weyl law*

$$N(\lambda) \sim \frac{\mathcal{L}^2}{4\pi} \lambda , \quad \lambda \rightarrow \infty ,\tag{3.24}$$

where  $\mathcal{L} = \sum_{e=1}^E l_e$  is the total length of the graph.

*Proof.* The Hilbert space  $H^1(D_\Gamma^*)$  is compactly embedded in  $L^2(D_\Gamma^*)$ . Since the form norm  $\|\cdot\|_{Q^{(2)}}$  is equivalent to the  $H^1(D_\Gamma^*)$ -norm, the Hilbert space  $(\mathcal{D}_{Q^{(2)}}, \|\cdot\|_{Q^{(2)}})$  is also compactly embedded in  $L^2(D_\Gamma^*)$ . Hence the operator associated with the quadratic form has compact resolvent [Dob05].

The Weyl law follows from a standard bracketing argument [RS78] based on a comparison with two suitable operators (quadratic forms), see also [BE09, BK11].

The first operator,  $(-\Delta_2, \mathcal{D}_2(P_D, L_D))$ , is the Dirichlet-Laplacian, and is characterised by the projector  $P_D = \mathbb{1}$  as well as  $L_D = 0$ . Given an operator  $(-\Delta_2, \mathcal{D}_2(P, L))$  the second comparison operator,  $(-\Delta_2, \mathcal{D}_2(P_R, L_R))$ , is the Laplacian given by the projector  $P_R = 0$  as well as

$$L_R = \text{diag}(\underbrace{\lambda, \dots, \lambda}_{4E^2\text{-times}}, \underbrace{0, \dots, 0}_{2E\text{-times}}), \quad (3.25)$$

where  $\lambda = \|\Lambda\|_{\text{op}}$ . The associated quadratic forms therefore satisfy the following inclusions of their domains,

$$\mathcal{D}_2(P_D, L_D) \subseteq \mathcal{D}_2(P, L) \subseteq \mathcal{D}_2(P_R, L_R). \quad (3.26)$$

Hence [RS78], it follows that the related eigenvalue-counting functions satisfy

$$N_D(\lambda) \leq N(\lambda) \leq N_R(\lambda). \quad (3.27)$$

As both  $N_D$  and  $N_R$  satisfy the Weyl law (3.24), the same asymptotics hold for  $N(\lambda)$ .  $\square$

## 4 Contact interactions for bosons

So far we assumed to have two non-identical particles on a graph. We now implement an exchange symmetry for two identical bosonic particles. Their states are described in the symmetric two-particle Hilbert-space  $\mathcal{H}_{2,B} = \mathcal{H}_1 \otimes_s \mathcal{H}_1$ . The orthogonal projection  $\Pi_B$  from  $\mathcal{H}_2 = L^2(D_\Gamma^*)$  to the bosonic subspace

$$L_B^2(D_\Gamma^*) = \Pi_B L^2(D_\Gamma^*) \quad (4.1)$$

acts on components of  $\Psi = (\psi_{e_1 e_2})$  with  $e_1 \neq e_2$  as

$$(\Pi_B \Psi)_{e_1 e_2}(x_{e_1}, y_{e_2}) = \frac{1}{2}(\psi_{e_1 e_2}(x_{e_1}, y_{e_2}) + \psi_{e_2 e_1}(y_{e_2}, x_{e_1})), \quad (4.2)$$

whereas on the components with  $e_1 = e_2$  its action reads

$$(\Pi_B \Psi)_{ee}^\pm(x_e, y_e) = \frac{1}{2}(\psi_{ee}^\pm(x_e, y_e) + \psi_{ee}^\mp(y_e, x_e)). \quad (4.3)$$

Due to this symmetry it would be sufficient to keep only components with  $e_1 < e_2$  in addition to the diagonal components with  $e_1 = e_2$ . For simplicity, when comparing to the previous section we, however, keep all components. We then denote the images of function spaces under the projection to their bosonic subspaces as, e.g.,

$$H_B^m(D_\Gamma^*) = H^m(D_\Gamma^*) \cap \mathcal{H}_{2,B}. \quad (4.4)$$

We note that whenever  $\Psi \in \mathcal{H}_{2,B}$  is in  $H^2(D_\Gamma^*)$ , the underlying symmetry implies the relations

$$\psi_{e_1 e_2, x}(x_{e_1}, y_{e_2}) = \psi_{e_2 e_1, y}(y_{e_2}, x_{e_1}) \quad \text{and} \quad \psi_{e_1 e_2, xx}(x_{e_1}, y_{e_2}) = \psi_{e_2 e_1, yy}(y_{e_2}, x_{e_1}) , \quad (4.5)$$

when  $e_1 \neq e_2$ , as well as

$$\psi_{ee, x}^\pm(x_e, y_e) = \psi_{ee, y}^\mp(y_e, x_e) \quad \text{and} \quad \psi_{ee, xx}^\pm(x_e, y_e) = \psi_{ee, yy}^\mp(y_e, x_e) . \quad (4.6)$$

Due to the bosonic symmetry it is possible to reduce the number of components in the vectors of boundary values (3.3). When  $e_1 \neq e_2$ , it suffices to keep the upper two components in each of the vectors (3.4), whereas for  $e_1 = e_2$  we use

$$\psi_{ee, bv}(y) = \begin{pmatrix} \sqrt{l_e} \psi_{ee}^-(0, l_e y) \\ \sqrt{l_e} \psi_{ee}^+(l_e, l_e y) \\ \sqrt{l_e} \psi_{ee}^+(l_e y, l_e y) \end{pmatrix} \quad \text{and} \quad \psi'_{ee, bv}(y) = \begin{pmatrix} \sqrt{l_e} \psi_{ee, x}^-(0, l_e y) \\ -\sqrt{l_e} \psi_{ee, x}^+(l_e, l_e y) \\ \sqrt{2l_e} \psi_{ee, n}^+(l_e y, l_e y) \end{pmatrix} , \quad (4.7)$$

with  $y \in [0, 1]$ . The space of boundary values therefore has dimension  $n_B(E) = 2E^2 + E$ , and decomposes in analogy to (3.8).

We also need the bounded and measurable maps  $P, L : [0, 1] \rightarrow \mathcal{M}(n_B(E), \mathbb{C})$ , where

1.  $P(y)$  is an orthogonal projector,
2.  $L(y)$  is a self-adjoint endomorphism on  $\ker P(y)$ ,

for a.e.  $y \in [0, 1]$ . The space  $\mathbb{C}^{n_B(E)}$  of boundary values decomposes in the same way as (3.8), however, the edge- $e$  subspaces are now one-dimensional. This forces the equivalent of (3.12) to be a projector on  $\mathbb{C}$  and to take values

$$P_{\text{contact}, e}(y) \in \{0, 1\} . \quad (4.8)$$

Here  $P_{\text{contact}, e}(y) = 1$  corresponds to a Dirichlet condition in the point  $(l_e y, l_e y)$  along the diagonal, whereas  $P_{\text{contact}, e}(y) = 0$  imposes no condition. Hence, when  $\delta$ -type interactions are considered we choose  $P_{\text{contact}, e}(y) = 0$ , and in the case of hardcore-interactions  $P_{\text{contact}, e}(y) = 1$  is chosen. Likewise, the equivalent of (3.13) is

$$L_{\text{contact}, e}(y) = -\frac{1}{2} \alpha(y) \quad (4.9)$$

for  $\delta$ -interactions, and  $L_{\text{contact}, e}(y) = 0$  for interactions of hardcore type.

We can now set up the following quadratic form,

$$Q_{P, L}^{(2), B}[\psi] = 2 \langle \Psi_x, \Psi_x \rangle_{L_B^2(D_\Gamma^*)} - 2 \int_0^1 \langle \Psi_{bv}(y), L(y) \Psi_{bv}(y) \rangle_{\mathbb{C}^{n_B(E)}} dy , \quad (4.10)$$

with domain

$$\mathcal{D}_{Q^{(2), B}} = \{ \Psi \in H_B^1(D_\Gamma^*); P(y) \Psi_{bv}(y) = 0 \text{ for a.e. } y \in [0, 1] \} . \quad (4.11)$$

As this is the restriction of a quadratic form on  $L^2(D_\Gamma^*)$  to  $L_B^2(D_\Gamma^*)$ , all results of Section 3 carry over: Propositions 3.2 and 3.3 imply that the quadratic form is closed and semi-bounded; when  $P$  is of class  $C^1$  and the form is regular, the associated self-adjoint operator is the bosonic two-particle Laplacian  $-\Delta_{2,B}$  with domain

$$\begin{aligned} \mathcal{D}_{2,B}(P, L) := \{ \Psi \in H_B^2(D_\Gamma^*); \ P(y)\Psi_{bv}(y) = 0 \text{ and} \\ Q(y)\Psi'_{bv}(y) + L(y)Q(y)\Psi_{bv}(y) = 0 \text{ for a.e. } y \in [0, 1] \} . \end{aligned} \quad (4.12)$$

According to Theorem 3.4, when the conditions of that theorem are fulfilled the quadratic forms leading to  $\delta$ -type and hardcore-interactions are regular.

We remark that for  $\delta$ -interactions one can use the decomposition (3.8) of the space of boundary values and the explicit expression (4.9) to rewrite the quadratic form as

$$\begin{aligned} Q_{P,L}^{(2),B}[\psi] &= 2\langle \Psi_x, \Psi_x \rangle_{L_B^2(D_\Gamma^*)} - 2 \int_0^1 \langle \Psi_{bv,vert}(y), L_{vert}(y)\Psi_{bv,vert}(y) \rangle_{\mathbb{C}^{2E^2}} dy \\ &\quad + \sum_{e=1}^E \int_0^1 \alpha(y) |\sqrt{l_e} \psi_{ee}^+(l_e y, l_e y)|^2 dy . \end{aligned} \quad (4.13)$$

In the same way, the form domain takes the form

$$\mathcal{D}_{Q^{(2),B}} = \{ \Psi \in H_B^1(D_\Gamma^*); \ P_{vert}(y)\Psi_{bv,vert}(y) = 0 \text{ for a.e. } y \in [0, 1] \} . \quad (4.14)$$

Due to the bosonic projection  $\Pi_B$ , which commutes with any of the two-particle Laplacians, asymptotically half of the spectrum of a Laplacian is projected to the bosonic Hilbert space  $\mathcal{H}_{2,B}$ , so that Proposition 3.5 implies the Weyl law

$$N_B(\lambda) \sim \frac{\mathcal{L}^2}{8\pi} \lambda , \quad \lambda \rightarrow \infty , \quad (4.15)$$

for the asymptotics of the eigenvalue count restricted to  $\mathcal{H}_{2,B}$ .

Our goal now is to study bosonic many-particle systems on graphs. Eventually, these have to be described in the bosonic Fock space over the one-particle Hilbert space. Since it suffices, however, to consider each  $N$ -particle space separately, we here only consider a fixed particle number  $N$ . In that context we shall introduce two-particle interactions that are (formally) of the type,

$$H_N = -\Delta_N + \sum_{i < j} \alpha(x_i) \delta(x_i - x_j) . \quad (4.16)$$

Due to the bosonic symmetry, on suitable functions the quadratic form associated with such an operator will be

$$\begin{aligned} \langle \Psi, H_N \Psi \rangle_{\mathcal{H}_{2,B}} &= \langle \Psi, -\Delta_N \Psi \rangle_{\mathcal{H}_{2,B}} \\ &\quad + \frac{N(N-1)}{2} \sum_{e_2 \dots e_N} \int_0^{l_{e_2}} \dots \int_0^{l_{e_N}} \alpha(x_{e_2}) |\psi_{e_2 e_2 \dots e_N}(x_{e_2}, x_{e_2}, \dots, x_{e_N})|^2 dx_{e_N} \dots dx_{e_2} . \end{aligned} \quad (4.17)$$

From (4.16) and (4.17) one concludes that contact interactions involve boundary values along hypersurfaces that are characterised by the fact that two particles are at the same position.

The configuration space of  $N$  (distinguishable) particles is

$$D_{\Gamma}^N = \bigcup_{e_1 e_2 \dots e_N} D_{e_1 e_2 \dots e_N} , \quad (4.18)$$

where  $D_{e_1 e_2 \dots e_N} = (0, l_{e_1}) \times \dots \times (0, l_{e_N})$ . We stress that this notation includes cases where several particles are on the same edge, in which case the same edge appears repeatedly. The hyperplanes that determine contact interactions are characterised by equations  $x_e^i = x_e^j$ , meaning that particles  $i$  and  $j$  sit on the same position on edge  $e$ . In analogy to (3.1), in order to implement contact interactions we have to decompose  $D_{\Gamma}^N$  further; this involves all hyperrectangles  $D_{e_1 e_2 \dots e_N}$  that are composed of at least two coinciding edges.

Now assume that  $(n_1, \dots, n_E)$  is a partition of  $N$  such that there are  $n_e$  particles on edge  $e$ . Let  $\sigma \in S_N$  assigns labels to the  $N$  particles in such a way that  $\sigma(1), \dots, \sigma(n_e)$  label the particles on edge  $e$ , with coordinates  $x_e^{\sigma(1)}, \dots, x_e^{\sigma(n_e)}$ . Permutations  $\pi \in S_{n_1} \times \dots \times S_{n_E} \subset S_N$  of particle labels then leave the assignment to edges untouched, and there exists such a permutation with

$$x_e^{\pi(\sigma(1))} \leq \dots \leq x_e^{\pi(\sigma(n_e))} , \quad \forall e \in \{1, \dots, E\} . \quad (4.19)$$

These relations define a polyhedral subdomain of  $D_{e_1 \dots e_N}$ . Every other permutation  $\pi \in S_{n_1} \times \dots \times S_{n_E}$  will produce a copy of that polyhedral subdomain that emerges through reflections in a succession of boundary hyperplanes. We will enumerate these  $n_1! \dots n_E!$  subdomains as  $D_{e_1 \dots e_N}^{\eta}$ , with  $1 \leq \eta \leq n_1! \dots n_E!$ .

In analogy to (3.1) we now introduce the dissected hyperrectangles as the disjoint union

$$D_{e_1 \dots e_N}^* = \bigcup_{\eta} D_{e_1 \dots e_N}^{\eta} . \quad (4.20)$$

The  $N$ -particle Hilbert space  $\mathcal{H}_N$  for  $N$  (distinguishable) particles with contact interactions can then be defined as

$$L^2(D_{\Gamma}^{N*}) = \bigoplus_{e_1 e_2 \dots e_N} L^2(D_{e_1 e_2 \dots e_N}^*) . \quad (4.21)$$

The corresponding Sobolev spaces are defined in the same way. Note that on the right-hand side  $D_{e_1 e_2 \dots e_N}^* = D_{e_1 e_2 \dots e_N}$  when all edges in the definition of  $D_{e_1 e_2 \dots e_N}$  are distinct, i.e., when no two particles are on the same edge. With this proviso, we denote functions on  $D_{e_1 e_2 \dots e_N}^*$  by  $\psi_{e_1 \dots e_N}$ , which themselves consist of  $n_1! \dots n_E!$  components defined on the subdomains  $D_{e_1 \dots e_N}^{\eta}$ , compare (3.1) and below.

The projection  $\Pi_B$  from (4.21) to the bosonic  $N$ -particle Hilbert space  $L_B^2(D_{\Gamma}^{N*})$  then is given by

$$(\Pi_B \Psi)_{e_1 \dots e_N} = \frac{1}{N!} \sum_{\pi \in S_N} \psi_{\pi(e_1) \dots \pi(e_N)}(x_{\pi(e_1)}^{\pi(1)}, \dots, x_{\pi(e_N)}^{\pi(N)}) . \quad (4.22)$$

In analogy to (3.9) and (3.10), two-particle interactions of a  $\delta$ -type (4.16) require boundary values of functions  $\Psi \in H_B^1(D_\Gamma^{N*})$  and their normal derivatives along (internal) boundary hyperplanes of the dissected hyperrectangles  $D_{e_1 e_2 \dots e_N}^*$ . In addition, boundary conditions at vertices have to be implemented. For those purposes the most convenient expression for the quadratic form is an analogue of (4.13).

We first introduce the vectors of boundary values in vertices. Due to the bosonic symmetry these can be given in the form

$$\Psi_{bv,vert}(\mathbf{y}) = \begin{pmatrix} \sqrt{l_{e_2} \dots l_{e_N}} \psi_{e_1 \dots e_N}(0, l_{e_2} y_1, \dots, l_{e_N} y_{N-1}) \\ \sqrt{l_{e_2} \dots l_{e_N}} \psi_{e_1 \dots e_N}(l_{e_1}, l_{e_2} y_1, \dots, l_{e_N} y_{N-1}) \end{pmatrix}, \quad (4.23)$$

and

$$\Psi'_{bv,vert}(\mathbf{y}) = \begin{pmatrix} \sqrt{l_{e_2} \dots l_{e_N}} \psi_{e_1 \dots e_N, x_{e_1}^1}(0, l_{e_2} y_1, \dots, l_{e_N} y_{N-1}) \\ -\sqrt{l_{e_2} \dots l_{e_N}} \psi_{e_1 \dots e_N, x_{e_1}^1}(l_{e_1}, l_{e_2} y_1, \dots, l_{e_N} y_{N-1}) \end{pmatrix}, \quad (4.24)$$

where  $\mathbf{y} = (y_1, \dots, y_{N-1}) \in [0, 1]^{N-1}$ . On these (vertex related) boundary values the bounded and measurable maps  $P_{vert}, L_{vert} : [0, 1]^{N-1} \rightarrow M(2E^N, \mathbb{C})$  shall act, which are required to fulfil

1.  $P_{vert}(\mathbf{y})$  is an orthogonal projector,
2.  $L_{vert}(\mathbf{y})$  is a self-adjoint endomorphism on  $\ker P_{vert}(\mathbf{y})$ ,

for a.e.  $\mathbf{y} \in [0, 1]^{N-1}$ .

Boundary values on internal hyperplanes in the dissected hyperrectangles involve components  $\psi_{e_1 \dots e_N}^\eta$  with a pair of coinciding edges,  $e_i = e_j$ . Due to the exchange symmetry we can always arrange for these edges to be  $e_1$  and  $e_2 = e_1$ . Permuting a given pair  $(e_i, e_j)$  to  $(e_1, e_2)$ , however, involves a change of the associated domain  $D_{e_1 \dots e_N}^\eta$  to some other copy  $D_{e'_1 \dots e'_N}^\eta$ . This means that  $\psi_{e_1 \dots e_N}^\eta$  is replaced by  $\psi_{e'_1 \dots e'_N}^\eta$ .

In analogy to (4.13) the quadratic form we wish to set up is

$$\begin{aligned} Q_B^{(N)}[\Psi] &= N \sum_{e_1 \dots e_N} \int_0^{l_{e_1}} \dots \int_0^{l_{e_N}} |\psi_{e_1 \dots e_N, x_{e_1}}(x_{e_1}, \dots, x_{e_N})|^2 dx_{e_N} \dots dx_{e_1} \\ &\quad - N \int_{[0, 1]^{N-1}} \langle \Psi_{bv,vert}, L_{vert}(\mathbf{y}) \Psi_{bv,vert} \rangle_{\mathbb{C}^{2E^N}} d\mathbf{y} \\ &\quad + \frac{N(N-1)}{2} \sum_{e_2 \dots e_N} \int_{[0, 1]^{N-1}} \alpha(y_1) |\sqrt{l_{e_2} \dots l_{e_N}} \psi_{e_2 e_2 \dots e_N}(l_{e_2} y_1, \mathbf{l}\mathbf{y})|^2 d\mathbf{y}. \end{aligned} \quad (4.25)$$

For convenience we here used the notation  $\mathbf{l}\mathbf{y} = (l_{e_2} y_1, l_{e_3} y_2, \dots, l_{e_N} y_{N-1})$ .

This form shall be defined on the domain

$$\mathcal{D}_{Q_B^{(N)}} = \{ \Psi \in H_B^1(D_\Gamma^{N*}); P_{vert}(\mathbf{y}) \Psi_{bv,vert}(\mathbf{y}) = 0 \text{ for a.e. } \mathbf{y} \in [0, 1]^{N-1} \}. \quad (4.26)$$

Using this, we can readily establish the following statements. These are immediate generalisations of the corresponding statements, Propositions 3.2 and 3.3, for two bosons ( $N = 2$ ).

**Theorem 4.1.** *Let the maps  $P_{vert}, L_{vert} : [0, 1]^{N-1} \rightarrow M(2E^N, \mathbb{C})$  as well as the function  $\alpha : [0, 1] \rightarrow \mathbb{C}$  be bounded and measurable. Then:*

- (i) *The quadratic form  $Q_B^{(N)}$  defined on the domain  $\mathcal{D}_{Q_B^{(N)}}$  is closed and semi-bounded.*
- (ii) *If  $P_{vert}$  is of class  $C^1$  and the form is regular, the associated self-adjoint operator is the  $N$ -particle Laplacian  $-\Delta_N$  with domain*

$$\begin{aligned} \mathcal{D}_{N,B}(P, L) := \{ \Psi \in H_B^2(D_\Gamma^{N*}); P(\mathbf{y})\Psi_{bv}(\mathbf{y}) = 0 \text{ and} \\ Q(\mathbf{y})\Psi'_{bv}(\mathbf{y}) + L(\mathbf{y})Q(\mathbf{y})\Psi_{bv}(\mathbf{y}) = 0 \text{ for a.e. } \mathbf{y} \in [0, 1]^{N-1} \} . \end{aligned} \quad (4.27)$$

Here  $P = P_{contact} \oplus P_{vert}$  and  $L = L_{contact} \oplus L_{vert}$  refer to all boundary values.

The proof of this Theorem is an immediate extension of the proofs of Propositions 3.2 and 3.3 as well as of Theorem 3.4.

It is also immediately clear from the proof of Proposition 3.5 that any of the  $N$ -particle Laplacians  $-\Delta_N$  with repulsive contact interactions have compact resolvent and hence possess purely discrete spectra, accumulating only at infinity. Furthermore, the eigenvalue counting function (compare (3.23) and (4.15)) satisfies a Weyl law that follows from a bracketing argument in the same way as (3.24). For the case of  $N$  distinguishable particles the Weyl law is

$$N(\lambda) \sim \frac{\mathcal{L}^N}{(4\pi)^{N/2} \Gamma(1 + \frac{N}{2})} \lambda^{N/2}, \quad \lambda \rightarrow \infty. \quad (4.28)$$

This follows most easily from the lower bound given by the Dirichlet Laplacian as in (3.27). The bosonic case requires to desymmetrise the spectrum with respect to particle exchange symmetry; hence, the bosonic counting function is reduced by a factor of  $\frac{1}{N!}$ .

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